Math 259A Lecture 4 Notes

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1 Correspondence Between Homeomorphisms and C^*-Algebra Morphisms

1.1 Recap: Homeomorphism between X_M and $\mathrm{Spec}(x)$

Recall our results from last time.

Proposition 1.1. *Let M be a commutative Banach algebra (over $\mathbb{C}$). Then* $\mathrm{Spec}_M(x) = \mathrm{Spec}_{C(X_M)}(\Gamma(x))$.

Proposition 1.2. *Let M be a C^*-algebra, and let $M_0 \subseteq M$ be a sub C^*-algebra. Then* $\mathrm{Spec}_M(x) = \mathrm{Spec}_{M_0}(x)$.

Theorem 1.1 (Gelfand). *Let M be a commutative C^*-algebra.*

1. *If $\varphi \in X_M$ is a character, then $\|\varphi\| = 1$ and $\varphi = \varphi^*$.*
2. *$\Gamma : M \to C(X_M)$ is a $*$-algebra isomorphism.*

Proposition 1.3. *Let M be a C^*-algebra generated by $x \in M$ and 1.*[1] *Then $\Psi : X_M \simeq \mathrm{Spec}(x)$ via $\varphi \mapsto \varphi(x)$ is a homeomorphism of compact spaces.*

Remark 1.1. Note that $\varphi(x) = \Gamma(x)(\varphi)$.

Proof. The map is surjective and well-defined by the first proposition above. Also, Ψ is continuous. If $\Psi(\varphi_1) = \Psi(\varphi_2)$, then $\varphi_1(x) = \varphi_2(x)$. But this implies that $\varphi_1(x^*) = \varphi_2(x^*)$. So $\varphi_1 = \varphi_2$ on all of M, as x generates M. So Ψ is injective. $\square$

[1] Alternatively, we can say, “Let M_0 be the sub C^*-algebra of M generated by x and 1.

1.2 Correspondence between homeomorphisms and C^*-algebra morphisms

Remark 1.2. If $\Delta : Z \to Y$ is a map between compact spaces, then we get a map $\Delta^* : C(Y) \to C(Z)$ given by $\Delta^*(f) = f \circ \Delta$. The map Δ^* is a a $*$-algebra homomorphism.

Conversely, if $\theta : M \to N$ is a morphism of unital C^*-algebras, we can view $\theta : C(X_M) \to C(X_N)$. Then there is a canonical $\Delta : X_N \to X_M$ such that $\theta = \Delta^*$ as follows. If $\varphi : N \to \mathbb{C}$ is multiplicative, then $\varphi \circ \theta : M \to \mathbb{C}$ is multiplicative. So $\Delta(\varphi) = \varphi \circ \theta$ is a well-defined map $X_M \to X_N$. Then $\Delta^* = \theta$. We denote this Δ by θ_* (so $(\theta_*)^* = \theta$).

Moreover, θ is surjective if and only if θ_* is injective and is injective if and only if θ_* is surjective. Thus, θ is an C^*-algebra isomorphism if and only if θ_* is a homeomorphism.

This is very important! It says that any homeomorphism between compact spaces corresponds to a *-algebra morphism between C^*-algebras.

Proposition 1.4. *If θ is surjective, then θ_* is injective.*

Proof. Let $\varphi_1 \neq \varphi_2 \in X_N$. Then $\varphi_1 \circ \theta \neq \varphi_2 \circ \theta$. □

Proposition 1.5. *If θ is injective, then θ_* is surjective.*

Proof. We get that $\theta : M \to N$ is isometric, as $\|y^*y\|_M = \mathrm{Spec}_M(y^*y) = \mathrm{Spec}_N(y^*y) = \|y^*y\|_N$ since y^*y is self-adjoint; then the C^*-condition gives that $\|y\|_M = \|y\|_N$.

Take a $\varphi \in X_M$ and consider the corresponding maximal ideal $M_\varphi \subseteq M$. Then $N\theta(M_\varphi)$ is a closed proper ideal in N (proper because it does not contain 1). Any maximal ideal M' containing $N\theta(M_\varphi)$ has the property that its character $\varphi' = \varphi_{M'} \in X_N$ satisfies $\theta_*(\varphi') = \varphi$. □

1.3 Continuous functional calculus

Let's be a bit more clear about a point made last lecture, using this viewpoint we have established.

Remark 1.3. Let M be a commutative C^*-algebra generated by x (so x is normal). Then $M \simeq C(X_M)$ via Γ. Note that since $\varphi(x) = \Gamma(x)(\varphi)$, using this identification, the map $(\Psi^{-1}) * \circ\Gamma$ sends $x^n \mapsto (z \mapsto z^n)$ and $(x^*)^m \mapsto (\overline{z} \mapsto \overline{z}^m)$. So for $f \in C(\mathrm{Spec}(x))$, we can define $f(x) := ((\Psi^{-1}) * \circ\Gamma)^{-1}(f)$. This is called **continuous functional calculus** for normal elements in a C^*-algebra.